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Radiation from perfect mirrors starting from rest and accelerating forever and the black body spectrum

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Abstract

We address the question of radiation emission from a perfect mirror that starts from rest and follows the trajectory $z = -\ln(\cosh t)$ until $t \to \infty$. We show that a correct derivation of the black body spectrum via the calculation of the Bogolubov amplitudes requires consideration of the whole trajectory and not just of its asymptotic part.

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1. Introduction

In a companion paper (Calogeracos 2002), hereafter referred to as I, we addressed the question of emission of radiation from accelerated mirrors following prescribed relativistic asymptotically inertial trajectories. The fact that the mirror moved at uniform velocity enabled us to define *in* and *out* states and employ standard time-dependent perturbation theory. In the present paper we consider a perfect mirror starting from rest and accelerating for an infinite time along the trajectory

$$z = g(t) = -\frac{1}{\kappa}\ln(\cosh \kappa t). \tag{1}$$

The above problem has been considered extensively in the past, starting with the classic papers by Fulling and Davies (1976) and Davies and Fulling (1977) (DF in what follows). The problem is of interest because of its (alleged) connection to radiation emitted from a collapsing black hole (Hawking 1975, DeWitt 1975) and to the attendant thermal spectrum. DF have calculated the renormalized matrix element of the T_{uu} component of the energy momentum tensor, the latter being defined as

$$T_{uu} = (\partial_u \phi)^2$$

(see e.g. Birrell and Davies (1982), equation (4.16)). The result is that *asymptotically for* $t \to \infty$

$$\langle T_{uu} \rangle \to \frac{\kappa^2}{48\pi}. \tag{2}$$

To derive (2) DF have used the asymptotic expression for the trajectory

$$g(t) \to -t - A\,\mathrm{e}^{-2\kappa t} + B \tag{3}$$

where A, B are constants that can be readily determined from (1). Equation (2) shows that there is a constant energy flux at late times, which is interpreted (p 249 of DF) as being analogous to the thermal energy flux found by Hawking (1975) in the case of a black hole. DF also calculated the Bogolubov amplitude $\beta(\omega, \omega')$ (and thus $\alpha(\omega, \omega')$ as well) and the spectrum

$$n(\omega) = \int_0^\infty \mathrm{d}\omega' |\beta(\omega, \omega')|^2\,\mathrm{e}^{-a(\omega+\omega')} \tag{4}$$

(a is a convergence factor). They did find that the spectrum coincides with the black body spectrum, namely that

$$|\beta(\omega, \omega')|^2 = \frac{1}{2\pi\omega'}\frac{1}{\mathrm{e}^{2\pi\omega} - 1}. \tag{5}$$

In this paper we wish to re-examine the method used by DF to obtain the expressions for the Bogolubov amplitudes. Our findings are summarized in the concluding section, and there is no reason to anticipate them here. The motivation for the investigation is provided by the following observations. (a) The Bogolubov $\alpha(\omega, \omega')$ and $\beta(\omega, \omega')$ amplitudes are by definition time-independent quantities. Hence one should be sceptical as to the validity of using the limiting form (3) *ab initio* without justification. (b) A careful calculation reveals that a certain term is missing from the starting equation (2.10*a*) of DF. Quite apart from the trivial nature of the error, it is not clear that the omission does not undermine the validity of the final result (5).

As far as the physics of the problem is concerned one may well counter that in the context of mirrors acceleration continuing for an infinite time implies mathematical singularities and also entails physical pathologies associated, for example, with the infinite energy that has to be imparted to the mirror. In that sense the realistic problem is examined in I. In the present paper we ignore such questions, take the premises of the early papers on the subject for granted, and concentrate on the calculation of the Bogolubov amplitudes and of the spectrum of the emitted radiation. In what follows we will show that the asymptotic behaviour of $\beta(\omega, \omega')$ for large ω' is

$$\beta(\omega, \omega') \approx (\omega')^{-\frac{1}{2}} + O((\omega')^{-N}) \qquad (N > 1). \tag{6}$$

The ω' integration in (4) is then logarithmically divergent. The divergence signifies according to DF (p 250) the production of particles at a finite rate for an infinite time. A similar divergence has been noted by Hawking (1975, p 211) in the context of black holes and the interpretation offered is the same. One often refers to the ultraviolet divergence by saying that large ω' frequencies dominate. In that same asymptotic limit $n(\omega)$ takes the familiar form of the black body spectrum (times the logarithmically divergent factor). Our analysis emphasizes two points: (a) the thermal result depends crucially on the fact that asymptotically the trajectory tends to a null line (the line $v = \ln 2$ in the case of (1)) and (b) for a mirror starting from rest and forever accelerating one must consider the *whole* trajectory and *not* just the asymptotic portion of it. The truth of statement (a) is usually taken as common knowledge. However, statement (a) is also taken erroneously to imply the negative of (b).

Note: Notation and conventions follow that of I (and are largely in accordance with the papers cited in the introduction). In section 2 we briefly go through the standard notation so that the paper is self-contained.

2. The formalism for a perfect mirror accelerating forever

We introduce the usual light cone coordinates

$$u = t - z \qquad v = t + z. \tag{7}$$

The massless Klein–Gordon equation reads

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0. \tag{8}$$

Hence any function that depends only on u or v (or the sum of two such functions) is a solution of (8). Let

$$z = g(t) = -\ln(\cosh t) \tag{9}$$

be the mirror's trajectory. Note that for large t the trajectory assumes the asymptotic form

$$g(t) \approx -t - \mathrm{e}^{-2t} + \ln 2.$$

Details are given in appendix A of I; for the moment observe that the trajectory is asymptotic to the line $v = \ln 2$ (figure 3 of I). In terms of the u, v coordinates the equation (9) of the trajectory is written in the form

$$u = f(v) \tag{10}$$

or alternatively

$$v = p(u) \tag{11}$$

where the function f is the inverse of p. The construction of the functions f and p given by the equation of the trajectory $z = g(t)$ is explained in section 2 of I. In the particular case of (9) the function f is given by

$$\begin{aligned} f(v) &= 1 && v \leqslant 0 \\ f(v) &= -\ln(2 - \mathrm{e}^v) && 0 \leqslant v < \ln 2. \end{aligned} \tag{12}$$

For future use we quote

$$f'(v) = -\frac{\mathrm{e}^v}{2 - \mathrm{e}^v}. \tag{13}$$

We take everything to exist to the right of the mirror. One set of modes satisfying (8) and the boundary condition

$$\phi(t, g(t)) = 0 \tag{14}$$

is given by (see I, equation (4))

$$\varphi_\omega(u, v) = \frac{\mathrm{i}}{2\sqrt{\pi\omega}}(\exp(-\mathrm{i}\omega v) - \exp(-\mathrm{i}\omega p(u))). \tag{15}$$

Another set of modes satisfying the boundary condition is immediately obtained from (15)

$$\bar{\varphi}_\omega(u, v) = \frac{\mathrm{i}}{2\sqrt{\pi\omega}}(\exp(-\mathrm{i}\omega f(v)) - \exp(-\mathrm{i}\omega u)). \tag{16}$$

The modes $\varphi_\omega(u, v)$ of (15) describe waves incident from the right as is clear from the sign of the exponential in the first term; the second term represents the reflected part which has a rather complicated behaviour depending on the motion of the mirror. These modes constitute the *in* space and should obviously be absent before acceleration starts,

$$a(\omega)|0_{\rm in}\rangle = 0. \tag{17}$$

Similarly the modes $\bar{\varphi}_\omega(u, v)$ describe waves travelling to the right (emitted by the mirror) as can be seen from the exponential of the second term. Correspondingly the first term is complicated. Regarding the *out* modes (16) the following point, which will be useful in the evaluation of the matrix element $\beta(\omega, \omega')$ (21), ought to be noted. Recall the picturesque form of such a mode (figure 6 of I), where for a perfect mirror $\bar{R}_R = 1$, $\bar{T}_R = 0$. Clearly in the coordinate range $u \leqslant 0$, $v \leqslant 0$ the mirror is stationary and thus the modes trivialize. During the subsequent motion of the mirror the variable v in the back-scattered wave in (16) is restricted between 0 and ln 2. These remarks will reflect on the range of integration in (23).

The $\bar{\varphi}_\omega(u, v)$ modes define the *out* space and

$$\bar{a}(\omega)|0_{\rm out}\rangle = 0. \tag{18}$$

The state $|0_{\rm out}\rangle$ corresponds to the state where nothing is produced by the mirror. The two representations are connected by the Bogolubov transformation

$$\bar{a}(\omega) = \int_0^\infty {\rm d}\omega(\alpha(\omega, \omega')a(\omega') + \beta^*(\omega, \omega')a^\dagger(\omega')). \tag{19}$$

Using (19) and its Hermitian conjugate we may immediately verify that the expectation value of the number of excitations of the mode (ω) in the $|0_{\rm in}\rangle$ vacuum is given by

$$\langle 0_{\rm in}|N(\omega)|0_{\rm in}\rangle = \int_0^\infty {\rm d}\omega'|\beta(\omega, \omega')|^2. \tag{20}$$

The matrix element $\beta(\omega, \omega')$ is given by (Birrell and Davies 1982), equations (2.9) and (3.36)

$$\beta(\omega, \omega') = -{\rm i}\int {\rm d}z\, \varphi_{\omega'}(z, 0)\frac{\partial}{\partial t}\bar{\varphi}_\omega(z, 0) + {\rm i}\int {\rm d}z\, \left(\frac{\partial}{\partial t}\varphi_{\omega'}(z, 0)\right)\bar{\varphi}_\omega(z, 0). \tag{21}$$

The integration in (21) can be over any spacelike hypersurface. Since the mirror is at rest for $t \leqslant 0$ the choice $t = 0$ for the hypersurface is convenient. The in modes evaluated at $t = 0$ are given by the simple expression

$$\varphi_\omega(u, v) = \frac{\rm i}{2\sqrt{\pi\omega}}(\exp(-{\rm i}\omega v) - \exp(-{\rm i}\omega u)) \tag{22}$$

i.e. expression (15) with $p(u) = u$ (corresponding to zero velocity). The $\bar{\varphi}$ modes are given by (16) with f depending on the trajectory. Relation (21) is rewritten in the form (the endpoints of integration shall be stated presently)

$$\begin{aligned}\beta(\omega, \omega') = -{\rm i}\int {\rm d}z \frac{\rm i}{2\sqrt{\pi\omega'}}\{{\rm e}^{-{\rm i}\omega' z} - {\rm e}^{{\rm i}\omega' z}\}\frac{\omega}{2\sqrt{\pi\omega}}\{f'\,{\rm e}^{-{\rm i}\omega f} - {\rm e}^{{\rm i}\omega z}\}\\ +{\rm i}\int {\rm d}z \frac{\omega'}{2\sqrt{\pi\omega'}}\{{\rm e}^{-{\rm i}\omega' z} - {\rm e}^{{\rm i}\omega' z}\}\frac{\rm i}{2\sqrt{\pi\omega}}\{{\rm e}^{-{\rm i}\omega f} - {\rm e}^{{\rm i}\omega z}\}.\end{aligned} \tag{23}$$

Bearing in mind the comments preceding (18) regarding the range of integration, the above expression may be rearranged in the form

$$\begin{aligned}\beta(\omega, \omega') = \frac{1}{4\pi\sqrt{\omega\omega'}}\int_0^{\ln 2} {\rm d}z\{{\rm e}^{{\rm i}\omega' z} - {\rm e}^{-{\rm i}\omega' z}\}\{\omega'\,{\rm e}^{-{\rm i}\omega f} - \omega f'\,{\rm e}^{-{\rm i}\omega f}\}\\ + \frac{(\omega - \omega')}{4\pi\sqrt{\omega\omega'}}\int_0^\infty {\rm d}z\{{\rm e}^{{\rm i}\omega' z} - {\rm e}^{-{\rm i}\omega' z}\}\,{\rm e}^{{\rm i}\omega z}\end{aligned} \tag{24}$$

where the limits of integration are displayed. The first and second integral in the above relation will be denoted by $\beta_I(\omega, \omega')$ and $\beta_{III}(\omega, \omega')$, respectively (in accordance with the notation in I). Note that the second integration is of kinematic origin and independent of the form of the trajectory. We also quote the expression for the $\alpha(\omega, \omega')$ amplitude

$$\alpha(\omega, \omega') = \mathrm{i} \int_0^\infty \mathrm{d}z \varphi_{\omega'}(z, 0) \frac{\partial}{\partial t} \bar{\varphi}^*_\omega(z, 0) - \mathrm{i} \int_0^\infty \mathrm{d}z \left(\frac{\partial}{\partial t} \varphi_{\omega'}(z, 0) \right) \bar{\varphi}^*_\omega(z, 0). \tag{25}$$

Observe the unitarity condition (see I for details)

$$\int_0^\infty \mathrm{d}\tilde{\omega} (\alpha(\omega_1, \tilde{\omega}) \alpha^*(\omega_2, \tilde{\omega}) - \beta(\omega_1, \tilde{\omega}) \beta^*(\omega_2, \tilde{\omega})) = \delta(\omega_1 - \omega_2) \tag{26}$$

which is a consequence of the fact that the set of *in* states is complete. Note that identity (69) of I is not valid because the set of *out* states is not complete (in the present case where the trajectory has a v asymptote). On the question of completeness see the remarks preceding equation (35) of I.

Recall from I that we can introduce quantities $A(\omega, \omega')$, $B(\omega, \omega')$ that are analytic functions of the frequencies via

$$\alpha(\omega, \omega') = \frac{A(\omega, \omega')}{\sqrt{\omega \omega'}} \qquad \beta(\omega, \omega') = \frac{B(\omega, \omega')}{\sqrt{\omega \omega'}}. \tag{27}$$

The quantity $B(\omega, \omega')$ is read off (23) (and $A(\omega, \omega')$ from the corresponding expression for $\alpha(\omega, \omega')$). From the definitions of the Bogolubov coefficients, the explicit form (23) of the overlap integral and expressions (15) and (16) for the field modes one can deduce that

$$B^*(\omega, \omega') = A(-\omega, \omega') \qquad A^*(\omega, \omega') = B(-\omega, \omega'). \tag{28}$$

The above relations allow the calculation of $\alpha(\omega, \omega')$ once $\beta(\omega, \omega')$ is determined.

3. Comparison between realistic trajectories and trajectories accelerating forever

A mirror trajectory involving acceleration forever cannot be considered as realistic in contrast to an asymptotically inertial trajectory. The former type of trajectory is of interest in that it can be taken as a simplified analogue of black hole collapse. The two trajectories are radically different from a physical point of view and this is reflected in the form of their spectra. In I we considered a mirror starting from rest, accelerating along the trajectory (1) until a spacetime point P (labelled by a coordinate $v = r < \ln 2$ in (u, v) coordinates), and then continuing at uniform velocity. We showed that in that case the Bogolubov amplitude squared $|\beta(\omega, \omega')|^2$ behaves asymptotically for large ω' as $(\omega')^{-5}$. In this work we set out to show that if acceleration is to continue forever then $|\beta(\omega, \omega')|^2$ goes as $1/\omega'$ (see (6)). It is thus not the case that the second problem can be simply considered as the $r \to \ln 2$ limit of the first. Although this may not come as a surprise from a physical point of view, it is certainly interesting to see where the dichotomy occurs mathematically. To this end we return to the problem examined in I where the $\beta(\omega, \omega')$ amplitude is given by (24) above with $\ln 2$ replaced by ∞ (cf also equation (34) of I), with the understanding that the function $f(z)$ stands for $f_{\mathrm{acc}}(z)$ given by (12) above when $z < r$, and for $f_0(z)$ given by (13) of I when $z > r$ (in the last equation B stands for the velocity β_P at point P where acceleration stops and the constant C is adjusted so that the velocity be continuous at P; see equation (49) of I). We introduce the Fourier integrals

$$I_1 = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}\omega' z - \alpha z}\, \mathrm{e}^{-\mathrm{i}\omega f(z)} \tag{29}$$

$$
\begin{aligned}
I_2 &= \int_0^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}\omega' z - \alpha z} f'(z)\, \mathrm{e}^{-\mathrm{i}\omega f(z)} \\
I_3 &= \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\omega' z - \alpha z}\, \mathrm{e}^{-\mathrm{i}\omega f(z)} \\
I_4 &= \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\omega' z - \alpha z} f'(z)\, \mathrm{e}^{-\mathrm{i}\omega f(z)}
\end{aligned}
\tag{30}
$$

where α is a small convergence factor taken to zero at the end of the calculations. Thus the f-dependent part of amplitude (35) of I is a linear combination of the above four integrals. The treatment that follows is equivalent to that of section 3.1 of I, however, it makes clearer the contrast between the two cases (realistic trajectory versus one that accelerates forever).

To examine the asymptotic behaviour for large ω' of the above integrals we use simple integration by parts (see Bender and Orszag (1978), p 278) according to the formula

$$
\int_a^b \mathrm{d}z F(z)\, \mathrm{e}^{\mathrm{i}\omega' z} = \left[\frac{F(z)}{\mathrm{i}\omega'}\, \mathrm{e}^{\mathrm{i}\omega' z} \right]_a^b - \frac{1}{\mathrm{i}\omega'} \int_a^b \mathrm{d}z F'(z)\, \mathrm{e}^{\mathrm{i}\omega' z}
\tag{31}
$$

which works provided the quantities appearing in the right-hand side are well defined. To treat I_1 of (29) we split the integral $\int_0^\infty$ to $\int_0^r + \int_r^\infty$ and apply (31) twice. The endpoint contributions from infinity vanish due to the convergence factor. The endpoint contributions at $z = r$ cancel out in pairs: $f_0(r)$ cancels with $f_{\mathrm{acc}}(r)$ (both accompanied by a factor $1/\omega'$) and $f_0'(r)$ cancels with $f'_{\mathrm{acc}}(r)$ (both accompanied by a factor $(\omega')^{-2}$). The origin of these cancellations is a direct consequence of the nature of the trajectory considered. The first cancellation reflects the fact that the trajectory itself $u = f(v)$ is continuous, and the second reflects the continuity of the velocity at P (see equation (15) of I for the general connection between $f'(v)$ and velocity). Similarly endpoint contributions at $z = 0$ cancel with corresponding terms originating from a large ω' expansion of the second (trajectory independent) term in (24) on the same continuity grounds as above. Thus integral I_1 goes as $(\omega')^{-3}$ and observing the prefactors in (24) we deduce that its contribution to $\beta(\omega, \omega')$ goes as $(\omega')^{-5/2}$.

The integral I_2 given by (30) can be handled in a similar way. However, because of the presence of $f'(z)$ we can integrate by parts only once if the endpoint contributions are to cancel out (recall that $f_0''(r) = f''_{\mathrm{acc}}(r)$ would require continuity of the acceleration). Thus I_2 goes as $(\omega')^{-2}$. Because the corresponding prefactor in (24) is one power of ω' smaller, the final contribution to the amplitude is again of the order $(\omega')^{-5/2}$.

We return to (24) with the upper limit of the first integral now being equal to $\ln 2$. It is manifestly obvious that there is now no room for the subtle cancellations exhibited above. We can still try integration by parts to see whether we can possibly arrive at any conclusions. To this end consider (for example) I_2 (with the upper limit now equal to $\ln 2$). Recall that according to (12)

$$
\mathrm{e}^{-\mathrm{i}\omega f(z)} = \mathrm{e}^{-\mathrm{i}\omega \ln(2-\mathrm{e}^z)} = (2 - \mathrm{e}^z)^{-\mathrm{i}\omega}
$$

and according to (13)

$$
f'(z) = -\frac{\mathrm{e}^z}{2 - \mathrm{e}^z}.
$$

Then

$$
I_2 = -\int_0^{\ln 2} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\omega' z} \frac{\mathrm{e}^z}{(2 - \mathrm{e}^z)^{1+\mathrm{i}\omega}}.
$$

If we now attempt to apply property (31) the upper endpoint contribution diverges. Hence we are unable to extract a further power of $1/\omega'$. It is clear that to obtain an asymptotic estimate in the present problem we have to resort to other methods.

4. Calculation of the Bogolubov amplitudes

The strategy we adopt in handling (24) is as follows. The first integral will be evaluated via an asymptotic expansion in negative powers of ω', which will in fact show that the ω' integration in (4) is logarithmically divergent. The second integral $\beta_{III}(\omega, \omega')$ in (24) is readily evaluated (see also I)

$$\beta_{III}(\omega, \omega') = \frac{1}{4\pi \mathrm{i}\sqrt{\omega\omega'}} - \frac{1}{4\pi \mathrm{i}\sqrt{\omega\omega'}}(\omega - \omega')\zeta(\omega + \omega') \tag{32}$$

where the function ζ and its complex conjugate ζ^* are defined in Heitler (1954, pp 66–71):

$$\zeta(x) \equiv -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}\kappa x}\,\mathrm{d}\kappa = P\frac{1}{x} - \mathrm{i}\pi\delta(x). \tag{33}$$

As explained in I it is only the first term in (33) that is operative as far as the calculation of the $\beta(\omega, \omega')$ goes. (The δ proportional term is only relevant in the calculation of the $\alpha(\omega, \omega')$ amplitude via relations (27) and (28).) Thus asymptotically in the large ω' limit

$$\beta_{III}(\omega, \omega') \approx \frac{1}{2\pi \mathrm{i}\sqrt{\omega\omega'}}. \tag{34}$$

We turn to the first integral $\beta_I(\omega, \omega')$ in (24). Rather than dealing with four integrals we perform an integration by parts to get (24) in the form

$$\beta_I(\omega, \omega') = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_0^{\ln 2}\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}\omega f(z) - \mathrm{i}\omega' z} + \frac{1}{2\pi}\frac{1}{\sqrt{\omega\omega'}}\sin(\omega\ln 2)\,\mathrm{e}^{-\mathrm{i}\omega f(\ln 2)}. \tag{35}$$

This is the same integration by parts that was used to obtain (54) of I and also the one that is used in DF to go from their (2.10a) to (2.10b). The second term in (35) oscillates rapidly since the exponent tends to infinity. Thus the term tends distributionally to zero and it may be neglected as in DF (also it is one power of ω' down compared to the first term). Of course this term was kept in (47) of I since it gives a finite contribution for $r < \ln 2$ (recall that at $v = r$ the trajectory considered in I reverts to uniform velocity). So asymptotically we are entitled to write

$$\beta_I(\omega, \omega') \approx -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_0^{\ln 2}\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}\omega f(z) - \mathrm{i}\omega' z} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_0^{\ln 2}\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}\omega' z}(2 - \mathrm{e}^z)^{\mathrm{i}\omega}. \tag{36}$$

To bring the singularity to zero we make the change of variable

$$z = \ln 2 - \rho \tag{37}$$

and rewrite $\beta_I(\omega, \omega')$ in the form

$$\beta_I(\omega, \omega') \approx -\frac{2^{\mathrm{i}(\omega - \omega')}}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_0^{\ln 2}\mathrm{d}\rho\,\mathrm{e}^{\mathrm{i}\omega'\rho}(1 - \mathrm{e}^{-\rho})^{\mathrm{i}\omega}. \tag{38}$$

We isolate the integral

$$I \equiv \int_0^{\ln 2}\mathrm{d}\rho\,\mathrm{e}^{\mathrm{i}\omega'\rho}(1 - \mathrm{e}^{-\rho})^{\mathrm{i}\omega}. \tag{39}$$

To obtain the asymptotic behaviour of (39) we adopt the standard technique of deforming the integration path to a contour in the complex plane (see Bender and Orszag (1978), ch 6); see also Morse and Feshbach (1953, p 610) where a very similar contour is used in the study of the asymptotic expansion of the confluent hypergeometric. The deformed contour runs from 0 up the imaginary axis to $\mathrm{i}T$ (we eventually take $T \to \infty$), then parallel to the real axis from $\mathrm{i}T$ to $\mathrm{i}T + \ln 2$, and then down again parallel to the imaginary axis from

$\mathrm{i}T + \ln 2$ to $\ln 2$. The contribution of the segment parallel to the real axis vanishes exponentially in the limit $T \to \infty$. We thus get

$$I = \mathrm{i}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\omega' s}(1-\mathrm{e}^{-\mathrm{i}s})^{\mathrm{i}\omega} - \mathrm{i}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{\mathrm{i}\omega'(\ln 2+\mathrm{i}s)}(1-\mathrm{e}^{-\ln 2-\mathrm{i}s})^{\mathrm{i}\omega}$$
$$= \mathrm{i}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\omega' s}(1-\mathrm{e}^{-\mathrm{i}s})^{\mathrm{i}\omega} - \mathrm{i}2^{\mathrm{i}\omega'}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\omega' s}\left(1-\frac{\mathrm{e}^{-\mathrm{i}s}}{2}\right)^{\mathrm{i}\omega}. \tag{40}$$

In the limit of large ω' the main contribution to (40) comes from the $s \approx 0$ region. We expand $\mathrm{e}^{\mathrm{i}s}$ in powers of s. We thus approximate one factor of the first integrand in (40) by

$$1-\mathrm{e}^{-\mathrm{i}s} \approx \mathrm{i}s \qquad (\mathrm{i}s)^{\mathrm{i}\omega} = \mathrm{e}^{-\pi\omega/2}s^{\mathrm{i}\omega}$$

where we set

$$\mathrm{i}s = |s|\,\mathrm{e}^{\mathrm{i}\pi/2} \tag{41}$$

took the branch cut of the function $\rho^{\mathrm{i}\omega}$ to run from zero along the negative x axis, wrote $\rho^{\mathrm{i}\omega} = \exp(\mathrm{i}\omega(\ln\rho + \mathrm{i}2N\pi))$ and chose the branch $N = 0$. For the second integrand we get

$$\left(1-\frac{\mathrm{e}^{-\mathrm{i}s}}{2}\right)^{\mathrm{i}\omega} \approx \left(\frac{1}{2}+\frac{\mathrm{i}s}{2}\right)^{\mathrm{i}\omega}.$$

Thus

$$I = \mathrm{i}\,\mathrm{e}^{-\pi\omega/2}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\omega' s}s^{\mathrm{i}\omega} - \mathrm{i}2^{\mathrm{i}(\omega'-\omega)}\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\omega' s}$$
$$= \mathrm{i}\,\mathrm{e}^{-\pi\omega/2}\frac{\Gamma(1+\mathrm{i}\omega)}{(\omega')^{1+\mathrm{i}\omega}} - \frac{\mathrm{i}2^{\mathrm{i}(\omega'-\omega)}}{\omega'}. \tag{42}$$

We substitute (42) in (38) and get

$$\beta_I(\omega,\omega') \approx -\mathrm{i}\frac{2^{\mathrm{i}(\omega-\omega')}}{2\pi\sqrt{\omega\omega'}}(\omega')^{-\mathrm{i}\omega}\,\mathrm{e}^{-\pi\omega/2}\Gamma(1+\mathrm{i}\omega) + \frac{\mathrm{i}}{2\pi\sqrt{\omega\omega'}}. \tag{43}$$

Observe the crucial cancellation of the second term in (43) with (34), the end result being

$$\beta(\omega,\omega') \approx -\mathrm{i}\frac{2^{\mathrm{i}(\omega-\omega')}}{2\pi\sqrt{\omega\omega'}}(\omega')^{-\mathrm{i}\omega}\,\mathrm{e}^{-\pi\omega/2}\Gamma(1+\mathrm{i}\omega). \tag{44}$$

As already mentioned at the end of the previous section, having determined $\beta(\omega,\omega')$ allows one to determine $\alpha(\omega,\omega')$. This is the typical form of the $\beta(\omega,\omega')$ amplitude that leads to the thermal spectrum; see e.g. DF. Indeed by taking the modulus of (44), squaring and using the property

$$|\Gamma(1+\mathrm{i}y)|^2 = \pi y/\sinh(\pi y)$$

we get the black body spectrum (5).

5. Conclusion

The objective of the paper was to prove that the Bogolubov amplitude $\beta(\omega,\omega')$ has the asymptotic form (6) and that the radiation emitted has the spectrum of a black body. Before enlarging on the conclusions we should elucidate one technical point in connection to the classic paper by Davies and Fulling (1977). It seems that the term $\beta_{III}(\omega,\omega')$ is unaccountably missing from (2.10*a*) of DF. Its existence is necessary if the unitarity conditions are to be satisfied. The need for the β_{III} term may be seen in a trivial example based on the results of I.

Consider the limiting case of a mirror perpetually at rest. Then $\beta_I(\omega, \omega')$ ((52) of I) disappears (formally $r = 0$) and β_{III} is instrumental in cancelling β_{II} of (53) evaluated at $r = 0$, $\beta_P = 0$ so that the total emission amplitude β vanishes (as it has to). The omission has been pointed out by Walker (1985) who, however, did not pursue the matter any further. On the other hand the present derivation shows that its existence is *crucial* in cancelling the non-thermal part of the $\beta_I(\omega, \omega')$ term.

There are various arguments in the literature in favour of the black body spectrum in the case of trajectory (1). In the present paper we are concentrating on a proof based on the Bogolubov coefficients. These quantities are by definition time-independent, and in this context the question as to where and when the photons are produced simply does not arise. In the same vein it is totally arbitrary to assert from the start that one specific part of the trajectory (in the present case the one near the asymptote) is more important than other parts. It is certainly true that were it not for the singularity on the v asymptote the thermal spectrum would not arise. However, the main conclusion of this paper is that the correct derivation of the thermal result requires the consideration of the complete trajectory and not just of its asymptotic part. In technical terms the function $f(z)$ in (36) cannot be approximated by its asymptotic expression. These remarks are strengthened by (i) the aforementioned role of the missing term (which appears *as if* it originates at $t = 0$) and (ii) the failure of the short distance expansion from the v asymptote as demonstrated in the appendix. All this is in accordance with one's quantum mechanical intuition. One's classical instincts might dictate that, roughly speaking, the small amount of time spent near the origin would have an insignificant effect compared to the infinitely long time spent near the asymptote. However, such loose statements are misleading in connection with the quantum mechanical calculation of global (*time-independent)* quantities. Similarly, attempts to distinguish between 'transient' and 'steady state' radiation at the level of the α and β amplitudes are bound to fail; the emphasis in the literature on the importance of the asymptotic part of the trajectory has unfortunately led to such statements. Having said all that, one can certainly enquire about the matrix elements of *local* field quantities as is indeed done in DF in a most illuminating way. This is however quite distinct from the calculation of Bogolubov amplitudes.

Mention must be made of the work of Carlitz and Willey (1987) where a mirror accelerating from the infinite past to the infinite future is considered. The authors do get the black body spectrum, their amplitudes do satisfy the Bogolubov identities, and quite clearly the time $t = 0$ plays no special role in their problem which is quite different from ours and our comments do not apply to their work.

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Appendix. Short distance expansion from the horizon and large frequencies

Let us consider the integral (39) and expand in small ρ. To first order

$$I^{(1)} = \int_0^{\ln 2} \mathrm{d}\rho\, \rho^{\mathrm{i}\omega}\, \mathrm{e}^{\mathrm{i}\omega'\rho}. \tag{45}$$

The integral (45) can be performed exactly in terms of the confluent hypergeometric and the asymptotic limit of large ω' may be examined afterwards. Let us make the change of variable $\rho = t \ln 2$ in (45) and rewrite

$$I^{(1)} = (\ln 2)^{\mathrm{i}\omega+1} \int_0^1 \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega' t \ln 2} t^{\mathrm{i}\omega}$$
$$= (\ln 2)^{\mathrm{i}\omega+1} \frac{1}{\mathrm{i}\omega + 1} M(1 + \mathrm{i}\omega, 2 + \mathrm{i}\omega, \mathrm{i}\omega' \ln 2) \tag{46}$$

(where M is the confluent hypergeometric function). An alternative way of presenting the above result is in terms of the incomplete gamma function γ^* by exploiting the connection of the latter with the confluent (see e.g. Tricomi (1954))

$$\gamma^*(\alpha, z) = \frac{M(\alpha, \alpha + 1; -z)}{\Gamma(\alpha + 1)}. \tag{47}$$

We can now examine the asymptotic limit of (46) for large ω'. The asymptotic limit of the confluent $M(a, b, \mathrm{i}|z|)$ for large values of $|z|$ is given by item 13.5.1 of Abramowitz and Stegun (1972) ($z \equiv \mathrm{i}\omega' \ln 2$). In the case $b = a + 1$ some simplifications occur and we get

$$M(1 + \mathrm{i}\omega, 2 + \mathrm{i}\omega, \mathrm{i}|z|) \approx -(1 + \mathrm{i}\omega)\, \mathrm{e}^{\mathrm{i}|z|} \frac{\mathrm{i}}{|z|} + \mathrm{i}\Gamma(2 + \mathrm{i}\omega) \frac{\mathrm{e}^{-\frac{\pi\omega}{2}}}{|z|^{1+\mathrm{i}\omega}} \tag{48}$$

(other terms are down by higher powers of $1/|z|$). The second term of the above relation combined with the prefactors of (46) does feature the $\Gamma(1 + \mathrm{i}\omega)\, \mathrm{e}^{-\frac{\pi\omega}{2}}$ factor characteristic of the black body spectrum. However, the presence of the first term spoils the thermal result (note that both terms are of the same order in ω'). In other words the result (48) is the correct answer to (45), which in turn is the wrong approximation of the original amplitude. The error is hardly surprising, since the term β_{III} has been omitted and the term β_I has been approximated in a totally unsystematic way.

To demonstrate that expansion in powers of ρ is a non-starter we go one step further in the expansion of (39) retaining terms of order ρ^2. Denoting this second-order approximation to I by $I^{(2)}$ we get

$$I^{(2)} = -\frac{\mathrm{i}\omega}{2} \int_0^{\ln 2} \mathrm{d}\rho\, \rho^{\mathrm{i}\omega+1}\, \mathrm{e}^{\mathrm{i}\omega'\rho}. \tag{49}$$

The calculation of (49) proceeds along exactly the same lines as that of (46) and yields

$$I^{(2)} = -\frac{\mathrm{i}\omega}{2} (\ln 2)^{\mathrm{i}\omega+1} \frac{1}{\mathrm{i}\omega + 2} M(\mathrm{i}\omega + 2, \mathrm{i}\omega + 3, \mathrm{i}\omega' \ln 2). \tag{50}$$

In the $\omega' \to \infty$ limit

$$M(\mathrm{i}\omega + 2, \mathrm{i}\omega + 3, \mathrm{i}|z|) \approx -\mathrm{i}(2 + \mathrm{i}\omega) \frac{\mathrm{e}^{\mathrm{i}|z|}}{|z|} - \Gamma(3 + \mathrm{i}\omega) \frac{\mathrm{e}^{-\frac{\pi\omega}{2}}}{|z|^{2+\mathrm{i}\omega}}. \tag{51}$$

The second term in (51) ought to be neglected compared to the second term in (48) since the former is down by one power of ω'. The first term in (51) is of the same order in ω' as the two terms in (48) and thus ought to be retained. The pattern persists to all orders, and is due to the structure $M(\mathrm{i}\omega + n, \mathrm{i}\omega + n + 1, \mathrm{i}\omega' \ln 2)$ (n integer) of the confluent in the present problem (the second argument being equal to the first plus 1). Thus an expansion in powers of ρ gives contributions of the same order of magnitude (i.e. $O(1/\omega')$ and hence is of little use.

In retrospect it may seem quite surprising how one obtains the correct result starting with the wrong expression (45). Let us return to that expression which (as detailed above) is the first-order approximation in a short distance expansion from the asymptote. The integral is often handled as follows (see for example Birrell and Davies (1982), p 108). One rescales the variable and writes the integral in the form

$$I^{(1)} = (\omega')^{-\mathrm{i}\omega} \int_0^{\omega' \ln 2} \mathrm{d}\rho\, \rho^{\mathrm{i}\omega}\, \mathrm{e}^{\mathrm{i}\rho}. \tag{52}$$

One now simply sets $\omega' \ln 2 = \infty$, changes variable $\rho = \mathrm{i}\sigma$ and rotates in the complex plane to get (52) in the form

$$I^{(1)} \simeq \mathrm{e}^{-\frac{\pi\omega}{2}} (\omega')^{-\mathrm{i}\omega} \int_0^{\infty} \mathrm{d}\sigma\, \mathrm{e}^{-\sigma} \sigma^{\mathrm{i}\omega}. \tag{53}$$

Note that setting $\omega' \ln 2 = \infty$ certainly does *not* amount to a systematic expansion in $(\omega')^{-1}$. The σ integration yields $\Gamma(1 + \mathrm{i}\omega)$ and one thus obtains the form for the β amplitude leading to the black body spectrum. On the other hand the integral (45) can be performed exactly (see (46) above) and the asymptotic estimate for large ω' (recall that we are chasing the ultraviolet divergence) is given by (48). The reason for the discrepancy lies in the fact that one should first evaluate the integral in terms of the confluent and then take the $\omega' \to \infty$ limit rather than taking the limit first. This rotation in the complex plane stumbles upon the Stokes phenomenon for the confluent (different limits for $|z| \to \infty$ depending on $\arg z$)). In other words it appears that Birrell and Davies have made two self-cancelling mistakes (wrong approximation to the amplitude and wrong evaluation of the integral).

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